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Nonlinear resonance for the oscillator with a nonmonotonic dependence of eigenfrequency on energy

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An alternative type of nonlinear resonance is reported. Its paradoxical peculiarity is demonstrated: an exact resonance between an external periodic field and a free oscillation at some energy is not necessarily needed, unlike the conventional nonlinear resonance. Examples of physical systems are given.

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The concept of nonlinear resonance (NR) is used broadly in physics (see [1–3] and references therein). In particular, for the case of a one-dimensional nonlinear oscillator subject to a periodic field, it means that the response of the oscillator to a weak field can be strongly nonlinear (i.e., constrained vibrations can have a large amplitude) if there is an energy E_r at which the eigenfrequency is exactly equal to the field frequency: $\omega(E_r)=\omega_f$ [Fig. 1(a)]. In this case, free vibrations of the oscillator with the energy $E=E_r$ are coherent with field oscillations. Correspondingly, if their phases coincide or are shifted for π then the oscillator does not change its energy, on average. Rigorous analysis [1] shows that, in the general case, slow small oscillations of the phase shift and of the energy take place. They are described with the motion in the cosine potential of an auxiliary particle with the phase shift as the coordinate, the difference between the action and its resonant value as the momentum and the mass $M=[d\omega(E_r)/dE_r]^{-1}$. A similar description of NR can be done for the case of many-dimensional motion [1–3] (several weakly interacting oscillatory degrees of freedom).

Nonlinear resonance plays a basic role in the phenomenon of dynamical stochasticity in Hamiltonian systems: the action of quick oscillations (which are omitted in the above-mentioned potential description) onto slow motion near the separatrix gives rise to the effective randomization of motion [1–3].

In this work, an alternative type of nonlinear resonance which takes place for a broad class of oscillatory systems is reported. One of its most intriguing and paradoxical features is demonstrated: at some conditions, the nonlinear resonance (i.e., a strong nonlinear response to a weak periodic field) can take place even in the absence of the exact resonance between an external field and free oscillations.

We shall consider the nonlinear oscillator with the dependence of eigenfrequency on energy $\omega(E)$ possessing either a maximum or a minimum [see, e.g., Fig. 1(b)]. Models of this type can describe superconducting quantum interference devices (SQUID's) [4,5] [corresponding potentials are of the type $U(q)=\cos q+l(q-q_0)^2$], electric oscillating circuits with a battery [6,7], local and resonant vibrations in certain doped crystals [8] subject to a constant homogeneous field [7] [corresponding potentials are like $U(q)=q^2+q^4+Aq$], polymeric molecules [9], and others. Such models were investigated intensively in the past few years because of interesting fluctuational phenomena arising due to the small dispersion of eigenfrequency near the extremum: zero-dispersion peaks [5,9–11], noise-induced spectral narrowing [7], zero-dispersion stochastic resonance [12,13].

If the periodic field acts on the oscillator then its motion is described with such dynamic equations for the coordinate q and momentum p:

$$dp/dt=-dU/dq+h\cos(\omega_f t),\quad dq/dt=p \tag{1}$$

(we consider the system without friction in order to emphasize the main effect and to simplify its demonstration; the presence of a weak friction does not prevent NR from taking place).

Using the canonical transformation to the variables of action I and phase ψ [14], Eq. (1) can be written after some identical transformations [9] (cf. also [3]) as the following:

$$dI/dt=\omega^{-1}(E)ph\cos(\omega_f t),$$
$$d\psi/dt=\omega(E)-\omega(E)q_E h\cos(\omega_f t), \tag{2}$$

where $q_E\equiv\partial q/\partial E$,

$$q\equiv 2\sum_{n=0}^{\infty} q_n(E)\cos(n\psi),$$

$$p\equiv -2\omega(E)\sum_{n=0}^{\infty} nq_n(E)\sin(n\psi), \tag{3}$$

$$E=p^2/2+U(q),\quad I=\int_{U_{\min}}^{E} dE/\omega(E).$$

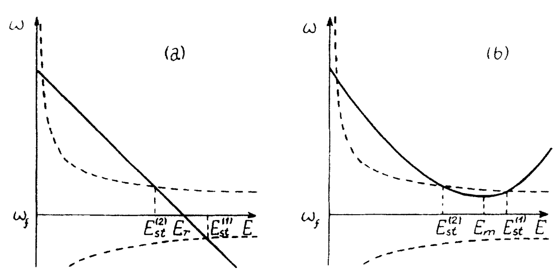

FIG. 1. Examples of a dependence of eigenfrequency on energy (solid lines): (a) monotonic, (b) nonmonotonic, and corresponding high-energy stationary states. Dotted lines correspond to functions $\omega_f\pm h\,dq_1/dI$.

Let the region of energies in which $|\omega(E)-\omega_f|\ll\omega_f$ exist. Then in this region, the so called slow phase $\tilde{\psi}\equiv\psi-\omega_f t$ changes slowly in the scale of $t\sim\omega_f^{-1}$ and we may average Eqs. (2) over high-frequency oscillations [15] that results in

$$dI/dt=-q_1 h\sin\tilde{\psi},$$
$$d\tilde{\psi}/dt=\omega-\omega_f-(dq_1/dI)h\cos\tilde{\psi}. \tag{4}$$

Let us find the stationary states for the system (4). $\tilde{\psi}_{\rm st}$ is obtained from the first equation: it is equal to either 0 or π. $I_{\rm st}$ should be found from

$$\omega(E(I_{\rm st}))-\omega_f=\pm h[dq_1(E(I_{\rm st}))/dI_{\rm st}], \tag{5}$$

where "+" corresponds to $\tilde{\psi}_{\rm st}=0$ and "−" corresponds to $\tilde{\psi}_{\rm st}=\pi$.

In the case of a conventional NR [see Fig. 1(a)] when the field frequency has the resonant eigenfrequency at some energy E_r, $\omega(E_r)=\omega_f$, and the field amplitude is small enough, there are two stationary states with the high energy (i.e., with $E\approx E_r$)

$$\tilde{\psi}_{\rm st}^{(1)}=\begin{cases}0, & I_{\rm st}^{(1)}=I_r+h(dq_1/dI)/(d\omega/dI)|_{I_r}\\ \pi, & I_{\rm st}^{(2)}=I_r-h(dq_1/dI)/(d\omega/dI)|_{I_r}\end{cases}$$
$$I_r\equiv I(E_r) \tag{6}$$

(just these states correspond to the NR while a low-value root of Eqs. (5) [see Fig. 1(a)] corresponds to small-amplitude vibrations [15], i.e., to the linear response which is not considered in the frame of a nonlinear resonance).

But we are interested in the different case: when the dependence of eigenfrequency on energy possesses either a maximum or a minimum while the field frequency does not have the resonant frequency among eigenfrequencies although it is close to the extremal one [see Fig. 1(b)]. In this case, there are, as a rule, two high-energy stationary states both of which have just the same phase [16]. The analysis of the linearized dynamic equations shows that, at small h, a stationary state is stable if

$$\mathrm{sgn}[(d\omega/dE)(dq_1^2/dE)(\omega-\omega_f)]>0 \tag{7}$$

and it is unstable at the opposite sign. That is, there are one stable and one unstable state in the majority of cases.

It could seem very strange that the stationary state with a large amplitude of vibrations exists notwithstanding that the difference between phases of the field and of any free vibration grows with time. But the point is that oscillations of the momentum caused by the field give rise to amplitude-modulated oscillations of the phase which can result in the compensation for the growth of the phase shift between the field and free vibrations [mathematically, it is expressed with the last term in the right hand part of the second equation (4)].

The difference between the conventional NR and the case discussed is not restricted to the different behavior at the absence of resonance between the field and free vibrations. Even when the resonant eigenfrequency exists but is close to the extremal eigenfrequency, the dynamics in the nonlinear resonance is completely different from the conventional NR. Unlike the latter case, the motion is not potential although still Hamiltonian:

$$H(I,\tilde{\psi})=\int dI(\omega-\omega_f)-hq_1\cos\tilde{\psi}. \tag{8}$$

The structure of the stochastic layer near the separatrix also is substantially different from the potential case.

We shall illustrate the aforesaid with an example. Let us consider the oscillator with the potential

$$U(q)=q^2/2+q^4/4+Aq. \tag{9}$$

Such a model describes electric oscillating circuits with a battery [6,7], local and resonant vibrations in certain doped crystals [8] subject to a constant homogeneous field [7]. At $|A|>A_c=8/(7)^{3/2}$, the function $\omega(E)$ possesses a minimum [7] like $\omega(E)$ shown in Fig. 1(b).

Let the field frequency coincide with the minimal eigenfrequency. Then allowing for a parabolic shape of the minimum, we obtain

$$\omega-\omega_f=\omega''\tilde{I}^2/2,$$

where

$$\omega''\equiv d^2\omega(E_m)/dE_m^2>0,$$
$$\tilde{I}\equiv I-I_m,\quad |\tilde{I}|\ll I_m. \tag{10}$$

Correspondingly the Hamiltonian is

$$H=\omega''\tilde{I}^3/6-hq_1\cos(\tilde{\psi}). \tag{11}$$

If the nonlinearity is weak for actual energies (i.e., $0<|A-A_c|\ll A_c$) then $q_1\approx(I/2)^{1/2}$ at $I\sim\leq I_m\equiv I(E_m)$. The dependence $q_1(E)$ is more complicated for a strongly nonlinear oscillator but, in any case, $dq_1/dI>0$. Allowing for this and Eqs. (5) and (10), we can find with an accuracy to the lowest power of h the stationary states:

$$\tilde{\psi}_{\rm st}=0,\quad \tilde{I}_{\rm st}=\pm(2hq_1'/\omega'')^{1/2},$$
$$q_1'\equiv dq_1(I_m)/dI_m, \tag{12}$$

where "+" corresponds to the stable states while "−" corresponds to the unstable states as follows from (7).

Substituting $\tilde{I}_{\rm st}$ of (12) corresponding to the unstable state into (11), we obtain the value of the energy for the motion along the separatrix and, thus, the equation of the separatrix:

$$\omega''\tilde{I}^3/6+h\{q_{1m}[1-\cos(\tilde{\psi})]-q_1'\,\tilde{I}\cos(\tilde{\psi})\}$$
$$=(hq_1')^{3/2}2(2/\omega'')^{1/2}/3, \tag{13}$$

where $q_{1m}\equiv q_1(I_m)$ and all powers of h higher than $h^{3/2}$ are omitted.

The typical picture of trajectories including the separatrix is shown in Fig. 2(b). It is seen from the comparison with Fig. 2(a) that it differs very much from the case of the conventional NR.

As concerns the stochastic layer near the separatrix, we have not enough room here to present its detailed rigorous

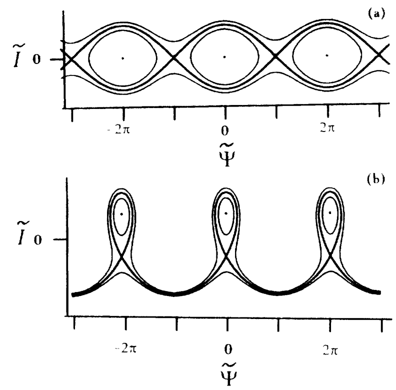

FIG. 2. Typical trajectories: (a) in the conventional NR; (b) in the zero-dispersion NR. Separatrixes are drawn by thicker lines. Stable states are indicated by dots.

analysis which will be done elsewhere [17]. We only notice here that, although the type of separatrix mapping is similar to the one for the conventional case, the coefficients which determine the width of the layer are different: in the asymptotic limit of a weak field, the layer is much narrower than in the conventional case. Besides this, the layer is not homogeneous unlike the conventional case: the width of the layer inside the separatrix loop [see Fig. 2(b)] is much narrower than outside it.

Thus for the discussed type of NR the behavior of the system differs very much from the conventional case. Therefore we propose to call the phenomenon with the special term the following: zero-dispersion nonlinear resonance (ZDNR), in analogy with the zero-dispersion peaks [5,9–11] and the zero-dispersion stochastic resonance [12,13], because the peculiarity of the phenomenon is determined just with the equality of the eigenfrequency dispersion in the extremum to zero, $d\omega(E_m)/dE_m=0$.

Let me conclude with several comments.

First, the phenomenon could be generalized for the case of many-dimensional motion when separate weakly interacting oscillatory degrees of freedom are characterized with minima and maxima of the dependences of partial eigenfrequencies on corresponding partial energies and there is not an exact resonance of the first order for partial eigenfrequencies. It should not be confused with a nonlinear resonance in so called intrinsically degenerate systems [18,19] (see also [1–3] and references therein) for which, in contrast to the conventional case, the resonance condition is satisfied in the whole energy space rather than in one point only.

Secondly, the phenomenon is not restricted with the parabolic minimum and maximum type of dependence of eigenfrequency on energy: any kind of nonmonotonicity and the inflection point as well results in completely different dynamics in comparison with the conventional nonlinear resonance.

Third, we expect that the most interesting applications of the zero-dispersion nonlinear resonance should be in SQUID's.

In summary, the theoretical prediction and the general description of the present type of nonlinear resonance have been reported. One of its paradoxical manifestations, the strong response of the oscillator to a weak periodic field at the absence of an exact resonance between the field and free oscillations, is demonstrated. The physical reason for the latter is the additional phase shift occurring due to the rapid oscillations of the momentum caused by the field.

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